Dirac fields in curved spacetime as Fermi-Hubbard model with non unitary tunnelings

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In this letter we show that a Dirac Hamiltonian in a curved background spacetime can be interpreted, when discretized, as a tight binding Fermi-Hubbard model with non unitary tunnelings. We find the form of the nonunitary tunneling matrices in terms of the metric tensor. In a simple case of a static diagonal metric, the tunnelings become unitary. Alternative interpretation of the Fermi-Hubbard Hamiltonian is that of a Pauli Hamiltonian, i.e. a non relativistic limit of the Dirac Hamiltonian. In this case the tunnelings remain, in general, non unitary even for the static diagonal metric. We discuss a possibility of synthesizing such Hamiltonians by means of laser assisted tunnelings in cold atomic experiments.

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Recently, much interest was devoted to a study of many body physics of quantum gases [1, 2]. High degree of control of the experimental parameters has allowed for designing specific Hamiltonians [3]. A special category is then a fabrication of synthetic gauge fields [4], where a remarkable experimental progress has been achieved in last couple of years, including the realization of synthetic electric [5] and magnetic [6] fields in the bulk as well as on the lattice [7]. A non abelian synthetic gauge field of the spin orbit Rashba type has been demonstrated in the bulk [8, 9]. In the case of a lattice, which will be of main interest, laser assisted hoppings [10, 11] allow for a simulation of a (non abelian) lattice gauge theory [12] with cold atoms [13]. Different works adressed the question of non abelian background fields with cold atoms. Such situation occurs e.g. in the case of electrons with spin orbit coupling, in both non interacting [14] and interacting [15] cases. In those scenarios, however, the tunnelings of different spin components between two adjacent lattice sites are described by unitary matrices, due to the hermiticity of the gauge field [12, 16]. Moreover, an explicit form of these matrices is determined by the theory one wants to simulate, e.g. the mentioned spin orbit coupling. It is thus an interesting question, what happens, if these tunnelings become non unitary, which can be, in principle, done in the cold atomic experiments using known techniques, as explained below.

Driven by this motivation, let us start with a kinetic fermionic Hamiltonian in $d-1$ spatial dimensions of a form

$$H(t) = \sum_{\mathbf{x},k,s,s'} \Psi_s^\dagger(\mathbf{x}) T_{s,s'}(x, x+a_k) \Psi_{s'}(\mathbf{x}+a_k) + \text{h.c.}, \quad (1)$$

where the sum runs over all lattice sites $\mathbf{x}$ and directions $k = 1..d-1$ and the fermionic operators satisfy the usual anticommutation relations

$$\{\Psi_s(\mathbf{x}), \Psi_{s'}^\dagger(\mathbf{x}')\} = \delta_{s,s'}\delta_{\mathbf{x},\mathbf{x}'}. \quad (2)$$

Throughout the paper, we denote the spacetime coordinates as x $(= (t, \mathbf{x}))$, while the space coordinates as $\mathbf{x}$. The matrices $T(x, x+a_k)$ represent a parallel transporter of a quantum field ψ between sites x and $x + a_k$. In the case of lattice gauge theories, the matrix T belongs to a representation of a gauge group, which is typically compact, such as $U(n)$ with n being the number of "flavor" components of the field ψ. In such case, the matrices T become unitary. The relevant question is thus, what if the T are non unitary? This question has actually been already adressed in the past [17–19] and more recently [20], but such interpretation seems to be problematic and was not actively pursued. Lets return to the Hamiltonian Eq.(1). In what follows, we will be interested in physics of fermions, so that ψ is a spinor. As discussed later, a general matrix $T \in Gl(n, \mathbb{C})$ can be engineered in cold atomic systems (n is the number of spin components). We would like to emphasize, that for such a novel situation, the Hamiltonian Eq.(1) is interesting on its own right. However, it is interesting to look, whether some physical significance can be given to it.

A starting point of our discussion will be a classical (field not quantized) fermionic field in a curved spacetime, which can be described by a Lagrangian density [21]

$$\mathcal{L}(x) = \sqrt{g}\left\{\frac{1}{2} i\bar{\psi}(x)\underline{\gamma}^\mu D_\mu\psi(x) + \text{h.c.} - m\bar{\psi}(x)\psi(x)\right\}. \quad (3)$$

Let us recall [22], that working in a coordinate basis e_μ, in which the spacetime vector x is defined in terms of its components x^μ, $x = x^\mu e_\mu$, one may construct a local orthonormal basis e_α. The two bases are related through vielbeins $e_\alpha = e_\alpha^\mu e_\mu$. The metric tensor g is defined as $\eta_{\alpha\beta} = e_\alpha^\mu e_\beta^\nu g_{\mu\nu}$, where η is the Minkowski, i.e. flat metric. Then, the curved spacetime γ matrices are defined as [23] $\underline{\gamma}^\mu = \gamma^\alpha e_\alpha^\mu$, where γ^α are the usual (flat spacetime) Dirac matrices, for which $\{\gamma^\alpha, \gamma^\beta\} = \eta^{\alpha\beta}$ and we adopt the sign convention $\eta = (+, -, -, ...)$. The covariant derivative acting on the spinor is $D_\mu = \partial_\mu - \Gamma_\mu$, where $\Gamma_\mu(x) =$

$\frac{1}{8}\left[\gamma^\alpha,\gamma^\beta\right]e^\nu_\alpha(x)(\nabla_\mu e_{\beta\nu})$ and $\nabla_\mu e_{\beta\nu} = \partial_\mu e_{\beta\nu} - \Gamma^\sigma_{\mu\nu}e_{\beta\sigma}$ (for a brief overview with essential technical details, see [24]). The canonically conjugate momentum to ψ can be found in a usual way as

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial(\partial_0\psi)} = \sqrt{g}\frac{1}{2}i\bar{\psi}\underline{\gamma}^0 \tag{4}$$

and similarly for $\bar{\pi}$ which is conjugate to $\bar{\psi}$. One then obtains the Hamiltonian density $\mathcal{H} = \pi(\partial_0\psi) + (\partial_0\bar{\psi})\bar{\pi} - \mathcal{L}$,

$$\mathcal{H}(x) = -\sqrt{g}\left\{\frac{1}{2}i\bar{\psi}\left[\underline{\gamma}^k D_k - \underline{\gamma}^0\Gamma_0\right]\psi + \text{h.c.} - m\bar{\psi}\psi\right\}. \tag{5}$$

Lets now cosider an isotropic square lattice in coordinate basis with lattice spacing a. We introduce a covariant derivative on the lattice as

$$D_\mu\psi(x) = \frac{1}{a}\left[P(x,x+a_\mu)\psi(x+a_\mu) - \psi(x)\right], \tag{6}$$

where $P(x,x+a_k)$ is the parallel transporter from $x+a_k$ to x and reads (here μ is fixed)

$$P(x,x+a_\mu) = \mathcal{P}\text{exp}\left[\int_{x+a_\mu}^{x}\mathrm{d}x^\mu\Gamma_\mu(x)\right] \tag{7}$$

and $\mathcal{P}$ stands for the path ordering. One can then formally discretize the Hamiltonian $H(t) = \int\mathrm{d}^{d-1}x\mathcal{H}(x)$,

$$H(t) = \frac{1}{2a}\sum_{\mathbf{x},k}\psi^\dagger(x)M^k P(x,x+a_k)\psi(x+a_k) \\ \psi^\dagger(x)\left[aM^0\Gamma_0 + \frac{1}{2}\sqrt{g}m\gamma^0\right]\psi(x) + \text{h.c.}, \tag{8}$$

where $M^\mu(x) \equiv -i\sqrt{g}\gamma^0\underline{\gamma}^\mu$. At first sight, the structure of the Hamiltonian Eq.(8) is similar to Eq.(1), but there are two major differences. First, in the former case, the fields are not quantized and second, they are time dependent, so it is not obvious, whether one can obtain the desired anticommutation relation Eq.(2) for the space dependent operators. In order to proceed, we shall rely on the arguments exposed in [25]. We summarize the main steps crucial for our purpose. The classical field $\psi(x)$ is a projection of ket $|\psi\rangle$ to a spatiotemporal basis $|x\rangle$

$$\psi(x) = \langle x|\psi\rangle. \tag{9}$$

Similarly, one obtains a different field which is only space dependent on some constant time hypersurface as

$$\Psi(\mathbf{x}) = \langle\mathbf{x}|\psi\rangle. \tag{10}$$

The relationship between the two fields $\psi(x)$ and $\Psi(\mathbf{x})$ can be found from the equivalence $\langle\phi|\psi\rangle = (\phi,\psi)$, where the scalar product in curved spacetime is defined as [25]

$$(\phi,\psi) = \int\mathrm{d}^{d-1}x\sqrt{g}\phi^\dagger\gamma^0\underline{\gamma}^0\psi. \tag{11}$$

From the resolution of identity $\int\mathrm{d}^{d-1}x\,|x\rangle\sqrt{g}\gamma^0\underline{\gamma}^0\langle x| = \mathbb{1}$ one obtains the evolution of the ket $|x\rangle$

$$\partial_0|x\rangle = -\frac{1}{2}|x\rangle(\partial_0\sqrt{g}\gamma^0\underline{\gamma}^0)(\sqrt{g}\gamma^0\underline{\gamma}^0)^{-1}, \tag{12}$$

which can be formally integrated to give

$$|x\rangle = |\mathbf{x},t\rangle = \sqrt{2}\,|\mathbf{x}\rangle\,(\sqrt{g}\gamma^0\underline{\gamma}^0)^{-\frac{1}{2}}. \tag{13}$$

The meaning of the factor $\sqrt{2}$ will become clear momentarily.

The quantization of the space and time dependent field Eq.(9), $\psi(x)$, then proceeds by imposing equal time anticommutation relations for the canonically conjugate operators [23], namely

$$\{\psi_s(x),\pi_{s'}(x')\} = i\delta(\mathbf{x}-\mathbf{x}')\delta_{s,s'}, \tag{14}$$

where $\pi(x)$ is given by Eq.(4). We can use the relations Eq.(9, 10) and Eq.(13) to find the relationship between the two fields $\psi(x)$ and $\Psi(\mathbf{x})$ to be

$$\psi(x) = \sqrt{2}\left(\sqrt{g}\gamma^0\underline{\gamma}^0\right)^{-\frac{1}{2}}\Psi(\mathbf{x}). \tag{15}$$

Plugging Eq.(4) into Eq.(14) and using Eq.(15) we obtain for the anticommutator of the constant time hypersurface fields

$$\{\Psi_s(\mathbf{x}),\Psi^\dagger_{s'}(\mathbf{x}')\} = \delta(\mathbf{x}-\mathbf{x}')\delta_{s,s'},$$

which is precisely the relation Eq.(2) (the factor $\sqrt{2}$ in Eq.(13) is actually an integration constant determined by the form of the anticommutator Eq.(2) and originates in the hermitian definition of the Lagrangian Eq.(3) containing factors 1/2).

Lets now take the lattice Dirac Hamiltonian Eq.(8) and write it as

$$H(t) = \sum_{\mathbf{x},k}\psi^\dagger(x)\tilde{T}(x,x+a_k)\psi(x+a_k) + \text{h.c.} + \psi^\dagger(x)\tilde{V}(x)\psi(x), \tag{16}$$

We now use the relation Eq.(15) to substitute for $\psi(x)$ and write the Hamiltonian as

$$H(t) = \sum_{\mathbf{x},k}\Psi^\dagger(\mathbf{x})T(x,x+a_k)\Psi(\mathbf{x}+a_k) + \text{h.c.} + \Psi^\dagger(\mathbf{x})V(x)\Psi(\mathbf{x}), \tag{17}$$

where

$$T(x,x+a_i) = \\ \frac{1}{2}(iM^0)^{-1/2}(x)M^i P(x,x+a_i)((iM^0)^\dagger)^{-1/2}(x+a_i) \\ V(x) = \\ (iM^0)^{-1/2}\left[M^0\Gamma_0 + (M^0\Gamma_0)^\dagger + \sqrt{g}m\gamma^0\right]((iM^0)^\dagger)^{-1/2} \tag{18}$$

where we have put the lattice spacing $a = 1$ for simplicity. Now, the Hamiltonian Eq.(17) has the same structure with the correct anticommutation relations for the

operators as Eq.(1) (plus the local term). The price to pay in order to achieve this goal was to absorb the spatiotemporal dependence of the fields $\psi(x)$ to the elements of the Hamiltonian and thus spoiling its covariance.

Next, we discuss a non relativistic limit of the Dirac equation. After all, the kinetic part of the usual Hubbard model for electrons in tight binding approximation (Eq.(1) with $T \propto \mathbb{1}$) is obtained from the non relativistic quantum mechanical Hamiltonian $H \propto \mathbf{p}^2/(2m)$. It is thus interesting to see, how similar derivation works for fermions in curved spacetime background. We should use a systematic method, known as Foldy-Wouthouysen transformation [26] (also used in the context of quantum fields in curved spacetimes [27]), which perturbatively decouples the electron and positron modes. One can derive the Dirac equation from Eq.(3)

$$(i\underline{\gamma}^\mu D_\mu - m)\psi(x) = 0, \tag{19}$$

which can be rewritten as Schrödinger equation

$$i\partial_0\psi(x) = \mathcal{H}_D\psi(x), \tag{20}$$

where

$$\mathcal{H}_D = (\underline{\gamma}^0)^{-1}(m - i\underline{\gamma}^k D_k) + i\Gamma_0 \tag{21}$$

is the Dirac Hamiltonian. The non relativistic limit can be obtained from the Dirac Hamiltonian of the form

$$\mathcal{H}_D = \gamma^0 m + \mathcal{E} + O, \tag{22}$$

where $\mathcal{E}$ and O are even and odd operator, defined by the property $\left[\gamma^0, \mathcal{E}\right] = 0$ and $\{\gamma^0, O\} = 0$ and γ^0 is in the Dirac representation. The lowest order expression for the nonrelativistic Hamiltonian is

$$\mathcal{H}_P = \gamma^0 m + \mathcal{E} + \frac{1}{2m}\gamma^0 O^2, \tag{23}$$

where the subscript P stands for the Pauli Hamiltonian. One can identify $\gamma^0, \mathcal{E}$ and O by comparing Eq.(22) with Eq.(21). In the most general case it yields rather lengthy expressions. In order to proceed with the calculation, we will thus consider a simple, yet non-trivial scenario with a static diagonal metric of the form

$$g = \begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix}, \tag{24}$$

and $h = \mathrm{diag}(h_{ii}(x^k))$, where $i = 1..d-1$ and the diagonal terms depend only on spatial coordinates x^k. First thing we note, is that in this case, the vielbein fields are also diagonal, $e^\mu_\alpha = 0$ for $\mu \neq \alpha$. In particular $e^0_0 = 1$ implying $(\underline{\gamma}^0)^{-1} = \underline{\gamma}^0 = \gamma^0$. Also, $\Gamma_0 = 0$. We then obtain for the Dirac Hamiltonian

$$\mathcal{H}_D = \gamma^0 m - i\gamma^0\underline{\gamma}^k D_k = \gamma^0 m + O, \tag{25}$$

since the term $\gamma^0\underline{\gamma}^k D_k$ is odd for the metric considered. We then obtain for the Pauli Hamiltonian density

$$\mathcal{H}_P = \gamma^0 m - \frac{1}{2m}\gamma^0(\gamma^0\underline{\gamma}^k D_k)(\gamma^0\underline{\gamma}^j D_j). \tag{26}$$

The total Hamiltonian, expressed in terms of field variables, then reads

$$H_P = \frac{1}{2}\left[(\psi, \mathcal{H}_P\psi) + (\mathcal{H}_P\psi, \psi)\right]. \tag{27}$$

The scalar product can be evaluated by integrating per parts in curved spacetime. The reason why one wants to do that is to obtain terms of type $(D\psi^\dagger)(D\psi)$ rather than $\psi^\dagger D^2\psi$, since the former can be mapped to a Hubbard model with only nearest neighbor hopping. Evaluating Eq.(27), we get

$$H_P = \frac{1}{2}\frac{1}{2m}\int \mathrm{d}^{d-1}x\sqrt{g}\bar{\psi}\underline{\gamma}^k\underline{\gamma}^j D_k D_j\psi + \mathrm{h.c.} + \int \mathrm{d}^{d-1}x\sqrt{g}m\bar{\psi}\psi \tag{28}$$

At this point ψ is still $2^{[d/2]}$ component spinor, where $[n]$ is the integer part of n. By construction, the Hamiltonian H_P contains only even operators and we can thus split the spinor into two parts, say $\psi = (\chi, \varphi)$. Each of the spinors χ, φ has $2^{[d/2]-1}$ components, which will have independent dynamics. In case of the diagonal static metric and $d = 4$, we find

$$\underline{\gamma}^k\underline{\gamma}^j D_k D_j = -\begin{pmatrix} e^k_k e^j_j \sigma^k\sigma^j\nabla_k\nabla_j & 0 \\ 0 & e^k_k e^j_j \sigma^k\sigma^j\nabla_k\nabla_j \end{pmatrix}, \tag{29}$$

where $\nabla_k = \partial_k - \tilde{\Gamma}_k$, $\tilde{\Gamma}_k = -1/4\sigma^j\sigma^l e^\nu_j(\mathbf{x})(\nabla_k e_{l\nu}(\mathbf{x}))\big|_{j<l}$. Lets write the Pauli Hamiltonian for one of the spinor components, say χ, which we write as

$$\int \mathrm{d}^3x\sqrt{g}\chi^\dagger\underline{\sigma}^k\underline{\sigma}^j\nabla_k\nabla_j\chi = \int \mathrm{d}^3\chi^\dagger f_{ii}\nabla_i\nabla_i\chi + \int \mathrm{d}^{d-1}\ \chi^\dagger f_{kj}\sigma^k\sigma^j[\nabla_k, \nabla_j]\big|_{k<j}\chi \tag{30}$$

where $f_{kj} = \sqrt{g}e^k_k e^j_j$. The commutator in the second term is familiar from non-abelian gauge theories and we have $[\nabla_k, \nabla_j] = \partial_{[j}\tilde{\Gamma}_{k]} - \left[\tilde{\Gamma}_k, \tilde{\Gamma}_j\right]$, which acts locally on the spinor χ. The first term can be integrated per parts to yield (using $\tilde{\Gamma}^\dagger_i = -\tilde{\Gamma}_i$)

$$\int \mathrm{d}^3x\chi^\dagger f_{ii}\nabla_i\nabla_i\chi = -\int \mathrm{d}^3x\left\{f_{ii}(\nabla_i\chi)^\dagger(\nabla_i\chi) - (\partial_i f_{ii})\chi^\dagger(\nabla_i\chi)\right\}. \tag{31}$$

We thus write the Pauli Hamiltonian as

$$H_P = \int \mathrm{d}^3x \frac{1}{2}\frac{1}{2m}\Big[f_{ii}(\nabla_i\chi)^\dagger(\nabla_i\chi) - (\partial_i f_{ii})\chi^\dagger(\nabla_i\chi) - \chi^\dagger f_{kj}\sigma^k\sigma^j[\nabla_k,\nabla_j]\big|_{k<j}\,\chi,\Big] + \text{h.c.} + \sqrt{g}m\chi^\dagger\chi. \tag{32}$$

We are now in the position to discretize the Pauli Hamiltonian, which is to follow exactly the same steps as in the case of Dirac Hamiltonian. Using again the prescription Eq.(15), which now takes a simple form $\chi(x) = \sqrt{2}\sqrt{g}^{-\frac{1}{2}}X(\mathbf{x})$, we arrive at a Hamiltonian, which can be formally written as Eq.(17), where we have to replace $\Psi \to X$ and the matrices T, V now depends only on spatial coordinates $\mathbf{x}$ and read

$$T(\mathbf{x},\mathbf{x}+a_i) = -\frac{\sqrt{g}^{-1}}{m}(f_{ii} + \frac{1}{2}\partial_i f_{ii})P(\mathbf{x},\mathbf{x}+a_i) \tag{33}$$

$$V(\mathbf{x}) = 2m + \frac{\sqrt{g}^{-1}}{m}\left[f_{ii} + f^-_{ii} - (\partial_i f_{ii}) + \left(\frac{1}{2}\sqrt{g}\,\underline{\sigma}^k\underline{\sigma}^j\,[\nabla_k,\nabla_j]\big|_{k<j} + \text{h.c.}\right)\right] \tag{34}$$

where $f^-_{ii} = f_{ii}(\mathbf{x}-a_i)$ and we have to replace $\Gamma \to \tilde{\Gamma}$ in the definition of the parallel propagator Eq.(7).

Physical interpretation; we just provided two possible interpretations of the kinetic Hamiltonian of the form Eq.(17), driven by a possible interpretation of non unitary T. However, it is interesting to notice, that for the static diagonal metric Eq.(24), $\underline{\gamma}$ are unitary, Γ_k are antihermitian, $\Gamma_k^\dagger = -\Gamma_k$ and the parallel propagators Eq.(7) become unitary. We thus have T which is also unitary, contrary to the Pauli Hamiltonian case. Another question is what field theory we are actually simulating. Lets take an example of simulation, where the Hamiltonian Eq.(1) describes a motion in a (two dimensional) plane for a two component field ψ_s, $s = 1, 2$. If we want to interpret it as a Dirac Hamiltonian, our simulator would correspond to a Dirac Hamiltonian in 2+1 dimensions. If we are to interpret it, however, as a Pauli Hamiltonian, the simulator corresponds to a spin half fermion living in 3+1 dimensions, but whose motion is confined to a plane.

Implementation with cold atoms; in the experiments with cold atoms, the internal degrees of freedom are usually played by the hyperfine states of the atoms. These allow for a laser assisted tunnelings between adjacent sites, say i, i' of the optical lattice. Lets denote the internal degrees of freedom s. In order to engineer an arbitrary $T(x) \in Gl(2,\mathbb{C})$, it is necessary to control each of the tunneling rates $(i,s) \leftrightarrow (i',s')$ independently in all spatial directions and moreover, the rates in general vary in spacetime. Different techniques and their combination can be used in order to achieve this goal. For example, bichromatic lattices can be combined with an independent Raman laser for each transition $s \leftrightarrow s'$ [28]. The spatial dependence is then given by a transverse profile of each Raman laser. It can be given e.g. by a (typically) gaussian laser profile which varies slowly on the lattice spacing or it can be designed using e.g. a specific phase masks [29] or array of microlenses [30], which allow for the modulation on the scale of lattice spacing and were already used in the cold atomic experiments. Another comment is, that the potential $V(x)$ in Eq.(17) is non diagonal and might be difficult to engineer. The way around is that since V is hermitian, it can be diagonalized by unitary transformation. It amounts to redefine the tunneling matrix T (analogous to a local gauge transformation in the case of gauge fields) and the spinors Ψ. Since the transformation is unitary, the anticommutation relations Eq.(2) are preserved.

Interactions; so far, we were talking about non interacting case. Although non abelian lattice gauge theories are non trivial already at this level, the most interesting physics can be obtained in the presence of interactions. A natural interaction term for spin half fermions in optical lattice is $H_{\text{int}} \propto U\sum_{s\neq s'} n_s n_{s'}$, where $n_s = \psi_s^\dagger\psi_s$ is the density operator. Once again, one entirely legitimate approach is to consider a Hamiltonian $H = H_{\text{kin}} + H_{\text{int}}$, with H_{kin} Eq.(1) and $T \in Gl(2,\mathbb{C})$ as such and study its properties (H could also describe interacting bosons instead of fermions). The other approach is to design it in a way, that it simulates a given field theory. For example, a proposal of simulation of a Thirring model (i.e. 1+1 dimensional field theory) with cold atoms was made recently [31], where the interaction term reads $J^\mu J_\mu$ with $J^\mu = \bar{\psi}\gamma^\mu\psi$. In curved spacetime, the replacement $\gamma^\mu \to \underline{\gamma}^\mu$ makes the interaction term spacetime dependent. One can thus try to modify the proposal [31] in a way that creates the correct interaction term, which might be an interesting test bed situation, since as one dimensional theory, the massless Thirring model is soluble also in curved spacetime [32].

We would like to mention, that a simulation of a Dirac field in curved spacetime with cold atoms was already adressed [33], but the discretization was carried out in the limit of small lattice spacing, such that the approximation $P \approx 1 + a\Gamma$ is valid. For stationary metrics, considered in [33], it results in unitary tunnelings.

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